CHAPTER 11

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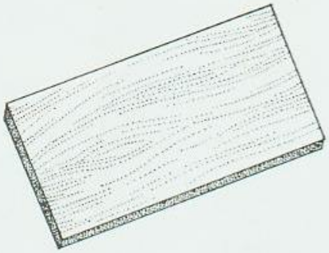
Area and Perimeter

11–1 Area Postulates

When a house is built, siding is nailed in place. Later it must be painted or stained. The roof is often covered with plywood sheets which must be covered with shingles. House construction provides many applications of the postulates and definitions introduced in this section.

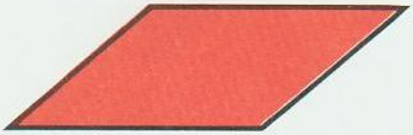


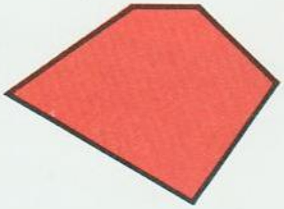
The edges of a sheet of siding represent a polygon called a rectangle. The surface of the sheet represents a subset of a plane called a *polygonal region.*

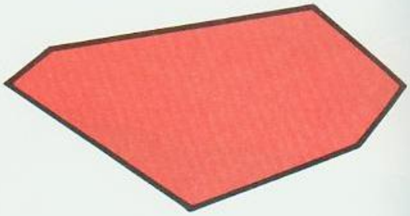
Definition 11–1

A **polygonal region** is a subset of a plane bounded by a polygon (or polygons).

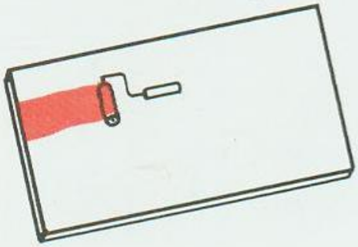
Here are three examples of polygonal regions.







How much stain is needed for a sheet of siding?

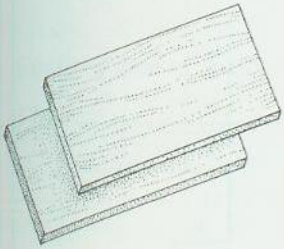


The amount of stain needed for a sheet of siding depends upon its size. A number called the *area* is used to describe this size.

Area Postulate

A unique positive number called the **area** can be assigned to each polygonal region. The area of region R is denoted by $A(R)$.

Properties of area are described by several postulates.

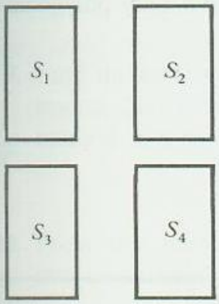


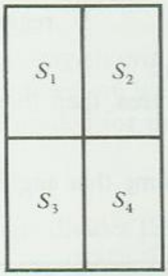
Two sheets the same size have the same area and should take the same amount of stain.

This fact is the subject of a postulate.

Area of Congruent Regions Postulate

If two rectangles or two triangles are congruent, then the regions they bound have the same area.





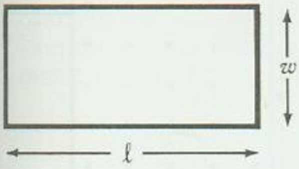
Four sheets of siding S_1, S_2, S_3, S_4 are often joined together.

The area of this four-piece section is equal to the sum of the areas of each piece. That is, $A(\text{4 pieces}) = A(S_1) + A(S_2) + A(S_3) + A(S_4)$.

Area Addition Postulate

If a polygonal region is the union of *n* non-overlapping polygonal regions, then its area is the sum of the areas of these *n*-regions.

To count the number of plywood sheets that are needed for a house, we need to be able to calculate the area of rectangular regions.



Area = ℓw

This last postulate in combination with the Area Postulate, the Area of Congruent Regions Postulate, and the Area Addition Postulate makes it possible to calculate area.

Rectangle Area Postulate

The area of a rectangle with length ℓ and width w is given by the formula ℓw.

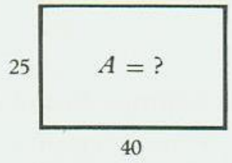
EXERCISES

A.

1. Draw a polygon and shade in the polygonal region determined by it.
2. Is the interior of a circle a polygonal region?
3. Is a rectangle a polygonal region?
4. If two rectangles have the same area, are they necessarily congruent?
5. Which postulate says that every polygonal region must have area?
6. Do the postulates say that only polygonal regions have area?
7. Draw a counterexample to this statement:
   If two polygonal regions have the same area, then they have the same number of sides.

Find the area of these regions. You may assume that angles appearing to be right angles are right angles.

8.



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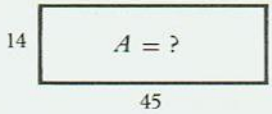

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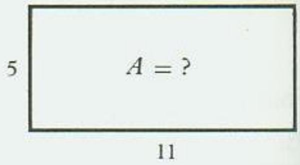
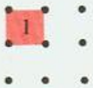
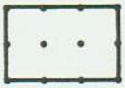


B.

For exercises 11–16 assume the area of the shaded area is 1. Find the area of each region by using the area postulates. (*Hint:* Look for rectangles and halves of rectangles.)



11.



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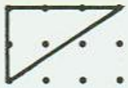

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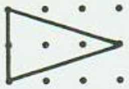
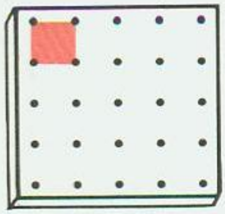


Activity

The area of the small shaded square on this geoboard is 1.

Construct on a geoboard or draw on dot paper triangles with areas of $\frac{1}{2}$, 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, and 3.



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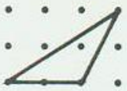

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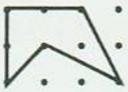
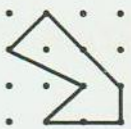
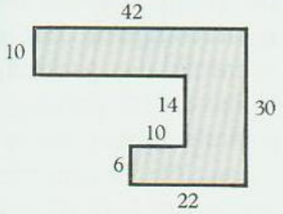


16.



Find the area of these regions.

17.



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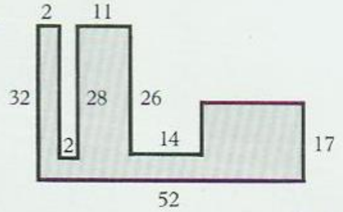
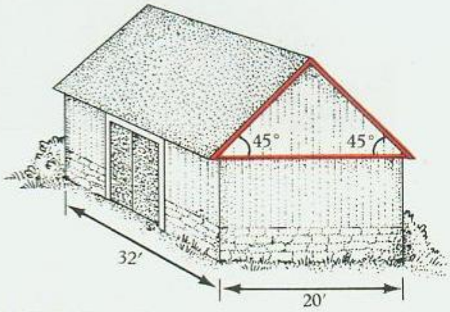


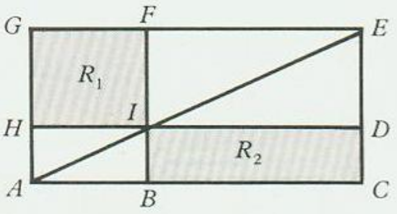
19. Find the area of the roof of the building shown. If you assume that 10% of the materials ordered are wasted, how many 4 × 8 sheets of plywood are needed for this roof?



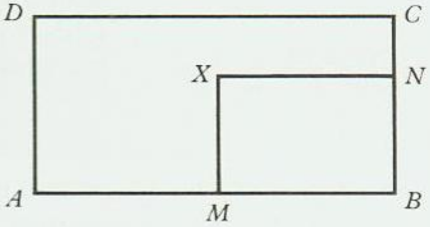
C.

20. Prove that the diagonal of a rectangle divides the rectangle into two triangles equal in area.

21. *ABIH*, *IDEF*, and *ACEG* are all rectangles. Explain why the areas of R_1 and R_2 are equal. (*Hint:* $A(\triangle ACE) = A(\triangle AGE)$.)

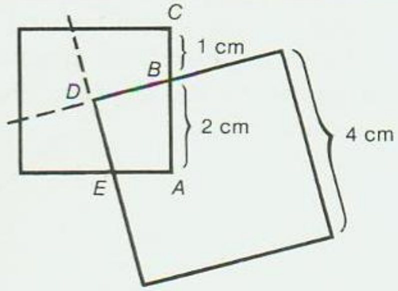


22. The area of *MBNX* is $\frac{1}{3}$ the area of *ABCD* and *M* is the midpoint of $\overline{AB}$. How does the length of $\overline{BN}$ compare to the length of $\overline{BC}$?

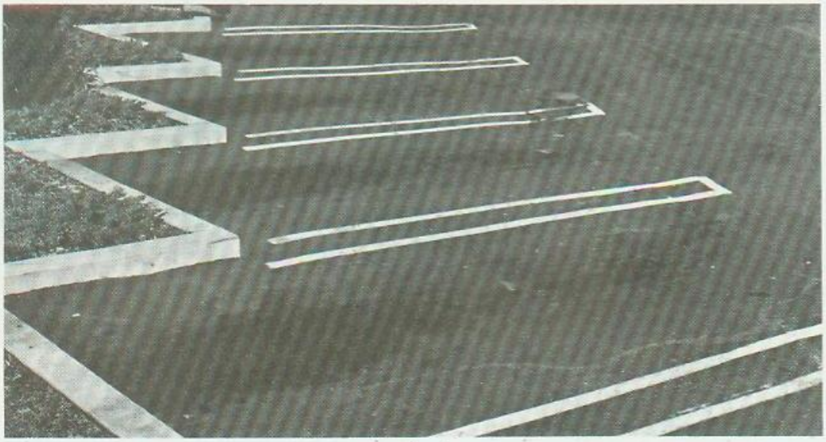


PROBLEM SOLVING

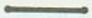
In the figure at the right, *D* is the center of the smaller of the two squares. The larger square overlaps the smaller square in quadrilateral *ABDE*. Find the area of this quadrilateral. Is it important that *B* trisects $\overline{AC}$?



11–2 Area of Parallelograms



There are situations in which it is important to find the area of regions that are not rectangular. For example, if a parking lot is to have slant parking, each parking spot is a parallelogram region. The amount of asphalt required for one spot depends upon the area of that parallelogram region.



1 centimeter

1 square centimeter

A unit of area must be chosen when measuring the area of regions like the one described above.

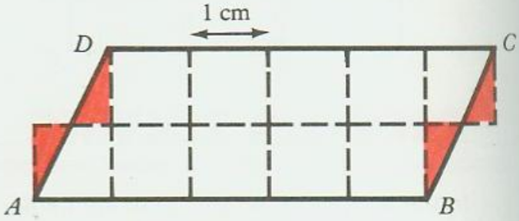
The most commonly used units of area are square inches, square feet, square yards, square centimeters, and square meters.

Definition 11-2

A **square unit** is a square region in which the length of a side is one unit of length.

The area of a region can be determined by counting the number of square units that are required to exactly cover the region.

By fitting together the square units and the congruent triangular regions, and using several area postulates, we conclude that the parallelogram on the right has an area of 10 square centimeters.

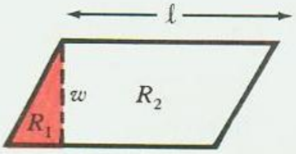


square unit

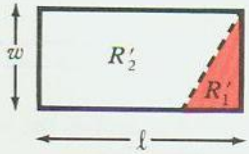
We write: $A(ABCD) = 10\ \text{cm}^2$

5

Another way to find the area of a parallelogram is to imagine that the triangular piece on one end is cut off and moved to the other end to form a rectangle. Using the Area of Congruent Regions Postulate and the Area Addition Postulate, we conclude that the areas of the parallelogram and rectangle are the same. Therefore, since the area of a rectangle is length times width, it follows that the area of a parallelogram is also "length" times "width."

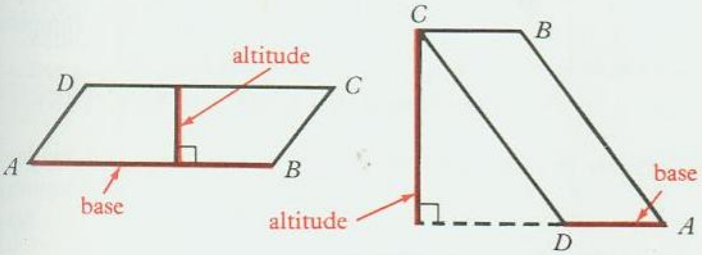


$R_1 \cong R_1'$
$R_2 \cong R_2'$



$A(R_1) + A(R_2) =$
$A(R_1') + A(R_2')$

In a parallelogram we will use the terms "base" and "altitude" instead of length and width. Any side of a parallelogram can be called the base. Once we choose a base, a segment perpendicular to that base, with endpoints on the base and its opposite side, is called a corresponding altitude. Notice that a parallelogram has two pairs of parallel bases.



Definition 11-3

An **altitude of a parallelogram** is a segment perpendicular to a pair of parallel sides with endpoints on those parallel sides. The **height** is the length of an altitude.

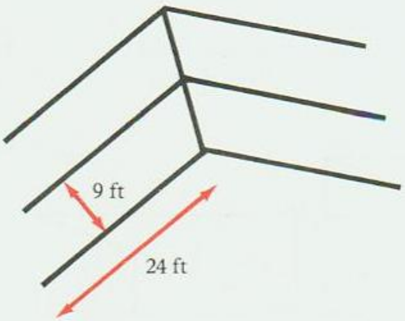
Theorem 11-1 Given a parallelogram with base b and corresponding height h, the area A is given by the formula $A = bh$.

APPLICATION

The standard slant parking spot is 9 feet wide and 24 feet long. What is the surface area of the asphalt that covers one parking spot?

Answer. A parking spot is a parallelogram with base 24 feet and altitude 9 feet. Applying Theorem 11-1 we calculate the area:

$$\text{area} = 24 \text{ ft} \times 9 \text{ ft} = 216 \text{ sq ft}$$

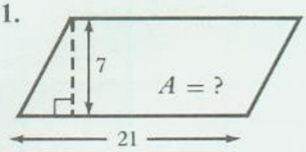


EXERCISES

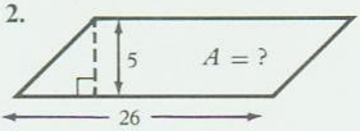
A.

In exercises 1–3, find the area of the parallelogram.

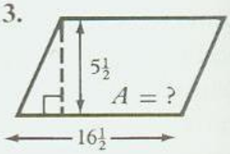
1.



2.

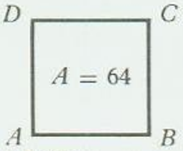


3.



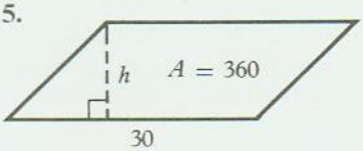
In exercises 4–6 the area of the parallelogram is given. Find the unknown part.

4.



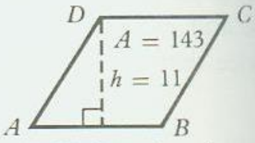
$ABCD$ is a square
$AB = ?$

5.



$h = ?$

6.

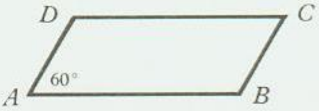


$ABCD$ is a rhombus
$AD = ?$

B.

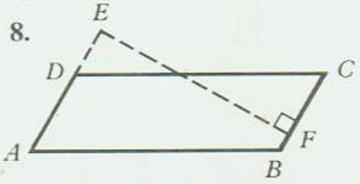
Find the area of each of the parallelograms in exercises 7–9.

7.



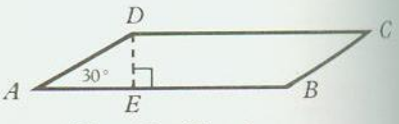
$AB = 14, AD = 5$

8.



$AD = 5, EF = 11$

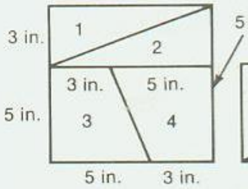
9.

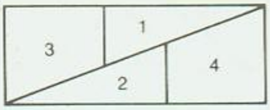


$AB = 12, AE = 5$

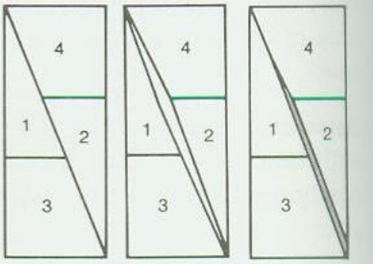
Activity

Draw this 8-inch square. Then cut and rearrange the pieces as shown.





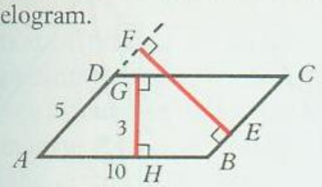
1. Which of the three situations indicated occurs?
2. Use area concepts to determine which one of the three actually happens.



"exact fit" "small gap" "small overlap"

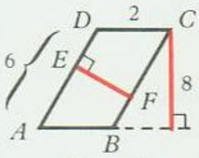
In exercises 10–12 find the unknown part. Each quadrilateral is a parallelogram.

10.



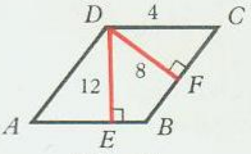
$EF = ?$

11.



$EF = ?$

12.

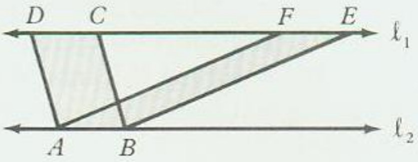


$AD = ?$

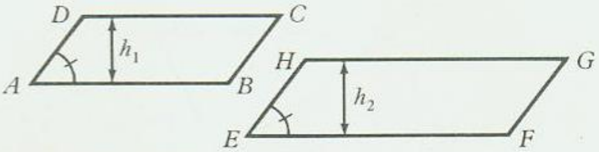
13. If the lengths of the sides of a parallelogram are doubled, what happens to the area of the parallelogram?

14. A parallelogram has sides of length 12 and 8 and one of the angle measures 120°. What is its area?

C.



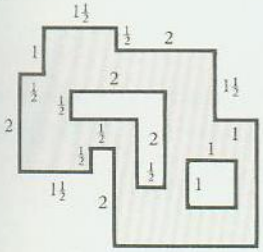
15. Lines ℓ_1 and ℓ_2 are parallel. Compare the area of parallelograms *ABEF* and *ABCD*.



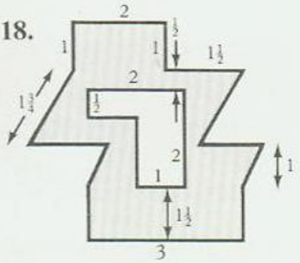
16. Given parallelograms *ABCD* and *EFGH* with $m\angle A = m\angle E$ and $h_2 = \sqrt{3}\,h_1$. If the area of *EFGH* is twice the area of *ABCD*, how does *AB* compare with *EF*?

Find the areas of the regions in exercises 17 and 18. Assume that segments that appear perpendicular or parallel are related in this way.

17.



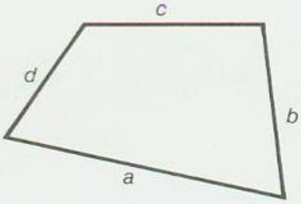
18.



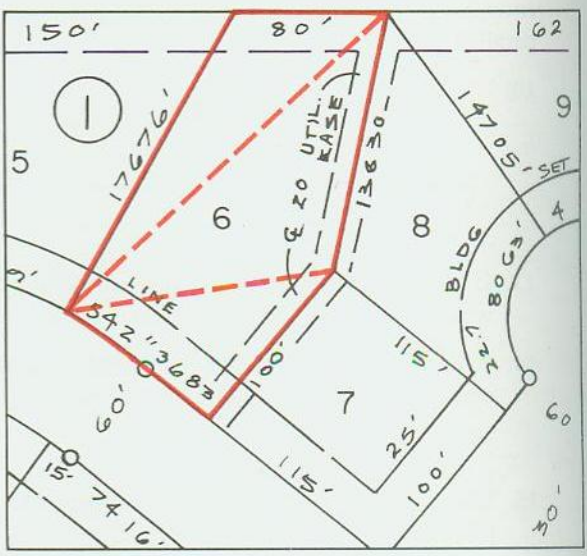
PROBLEM SOLVING

The ancient Babylonians used the formula $\dfrac{(a+c)(b+d)}{4}$ for the area of a quadrilateral with sides of length *a*, *b*, *c*, and *d*.

1. Does this formula work for rectangles?
2. Does this formula work for parallelograms?

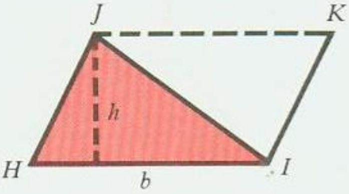


11–3 Areas of Triangles and Trapezoids

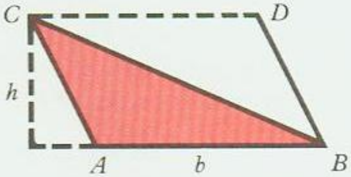


A civil engineer needs to find the area of an irregularly shaped lot in a housing subdivision like lot #6 shown here. This can be done by dividing the region into triangular regions and calculating the area of each triangular region.

The figures below illustrate that a triangular region may be thought of as one half of a parallelogram region. Therefore, the formula for finding the area of a parallelogram leads to an area formula for triangles.



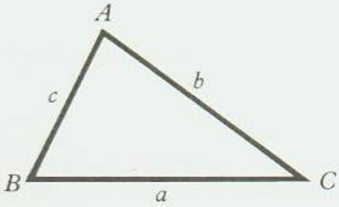
$$A(\triangle HIJ) = \tfrac{1}{2}A(HIKJ) = \tfrac{1}{2}bh$$



$$A(\triangle ABC) = \tfrac{1}{2}A(ABDC) = \tfrac{1}{2}bh$$

Theorem 11–2 Given a triangle with base b and corresponding altitude h, the area A is given by the formula $A = \frac{1}{2}bh$.

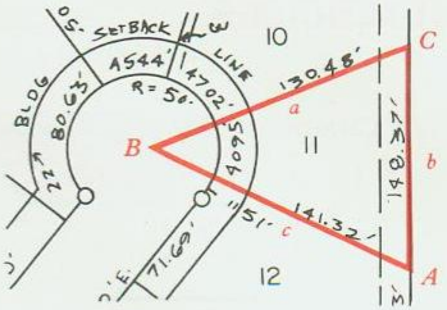
Sometimes the lengths of all three sides of a triangle may be known, but an altitude may not be. In that case a formula, known to Heron of Alexandria in the first century, is useful.



$s = \frac{1}{2}(a + b + c)$

Theorem 11–3 **Heron's Formula.** If $\triangle ABC$ has sides of length a, b, and c, then $A(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a + b + c)$.

APPLICATION



One method of finding the area of lot #11 requires finding the area of $\triangle ABC$ first. By using Heron's formula and a hand calculator, this is a manageable task. (Note that the radius of the circle is 50 feet.)

$a = 130.48 + 50 = 180.48 \qquad b = 148.37$

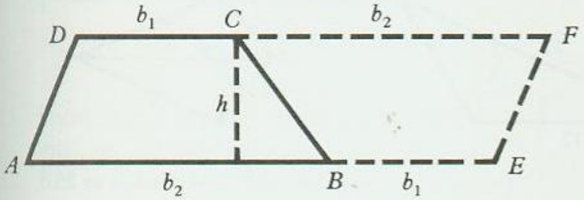
$c = 141.32 + 50 = 191.32$

$s = \frac{1}{2}(a + b + c) = 260.09$

$s - a = 79.61,\ s - b = 111.72,\ s - c = 68.77$

By Heron's Formula $A(\triangle ABC) \doteq \sqrt{(260.09)(79.61)(111.72)(68.77)} \doteq 12{,}613$.

A trapezoid may also be viewed as one half of a parallelogram. The area of the trapezoid is one half the area of that parallelogram.



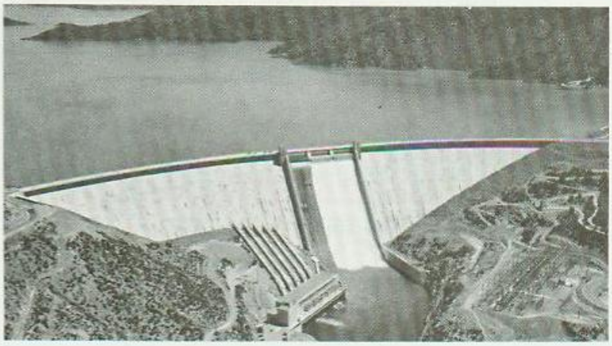
base of $\square AEFD = b_1 + b_2$
$A(\square AEFD) = h(b_1 + b_2)$

Theorem 11-4 Given a trapezoid with bases b_1 and b_2 and altitude h, the area A is given by the formula $A = \frac{1}{2}h(b_1 + b_2)$.

APPLICATION

A dam often has a trapezoidal cross section. The designer of a dam must determine the area of this trapezoidal cross section. If the dam is 180 meters tall and has bases 10 meters and 60 meters long, what is the area of the cross section? By using Theorem 11-4 we calculate the area:

$\text{area} = \frac{1}{2} \cdot 180(10 + 60)\ \text{m}^2 = 6300\ \text{m}^2$



EXERCISES

A.

Calculate the area of each region in exercises 1-11.

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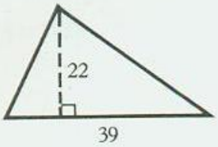

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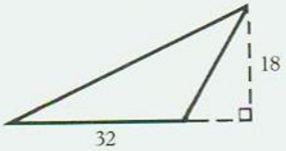

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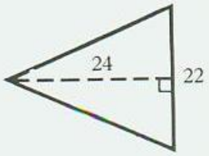

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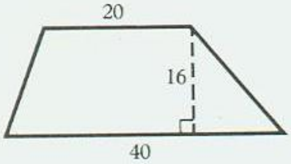

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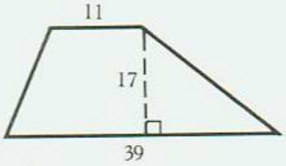

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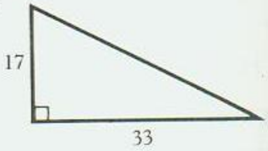

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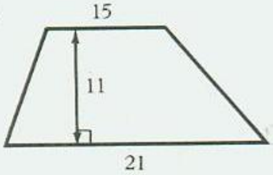

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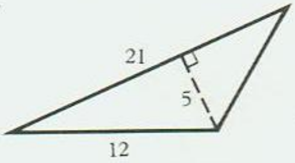

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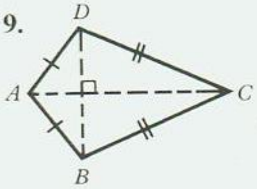
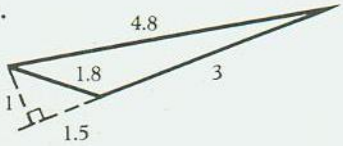
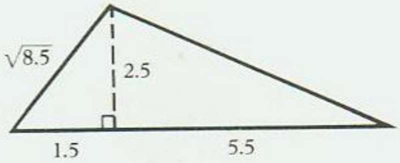


$AC = 41 \quad BD = 25$

10.



11.



B.

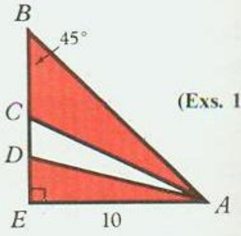
12. The area of $\triangle ABC$ is 48 cm² and $AB = 6$ cm. What is the length of the altitude on $\overline{AB}$?

13. A trapezoid with base lengths 14 ft and 21 ft has an area of $87\frac{1}{2}$ sq ft. What is its altitude?

14. What is the area of the shaded region?

15. What is the area of the unshaded region?

16. What is the ratio of the area of the shaded region to the area of $\triangle ABE$?

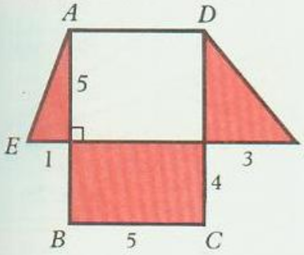


(Exs. 14-16)

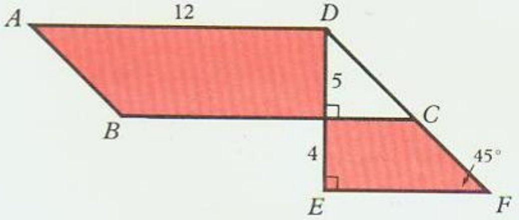
C is the midpoint of $\overline{BE}$.
D is the midpoint of $\overline{CE}$.

For exercises 17–18 find the area of the shaded region.

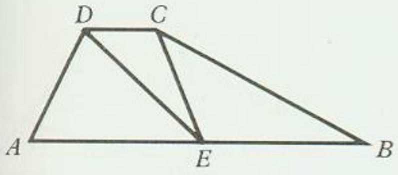
17. *ABCD* is a rectangle.



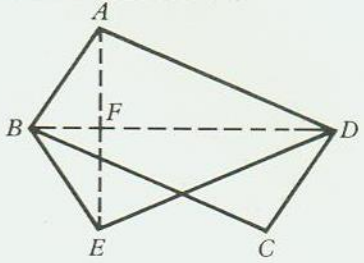
18. *ABCD* is a parallelogram.



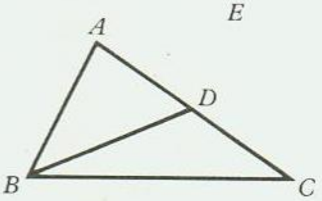
19. *ABCD* is a trapezoid and *E* is the midpoint of $\overline{AB}$. Show that the area of *AECD* is equal to the area of *EBCD*.



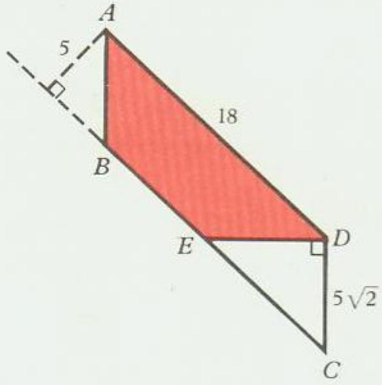
20. *ABCD* is a parallelogram whose area is 60 square units. Find the area of the kite *ABED*. (*Hint:* Find the area of △*ABD*.)



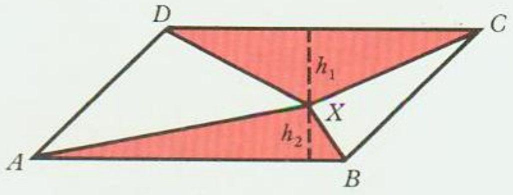
21. Show that a median divides a triangle into two regions of equal area. That is, if $\overline{BD}$ is a median of △*ABC*, show that the area of △*ABD* is equal to the area of △*BDC*.



22. *ABCD* is a parallelogram. Find the length *BE*.



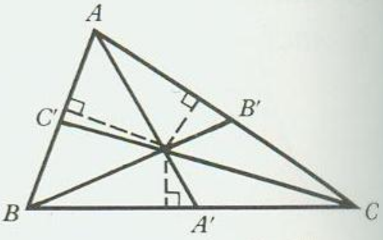
23. *ABCD* is a parallelogram and *X* is any point inside. Show that the area of the shaded region is one half the area of the parallelogram. (*Hint:* $h_1 + h_2$ is the altitude of *ABCD*.)



24. Use Heron's Formula to find the area of a triangle with sides of these lengths.

a. 3 cm, 4 cm, 5 cm **b.** 17 cm, 18 cm, 19 cm (Use a calculator.)

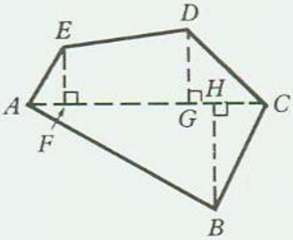
c. 25 cm, 36 cm, 41 cm (Use a calculator.)



(Ex. 25)

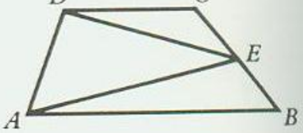
25. Show that the three medians of a triangle divide the triangle into six regions that are equal in area.

26. A developer bought a piece of land which formed an irregular pentagon. Find the area of this land if $AF = 10$ m, $FG = 40$ m, $GH = 15$ m, $HC = 20$ m, $EF = 20$ m, $DG = 30$ m, and $HB = 35$ m.

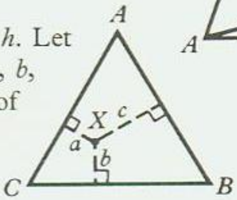


C.

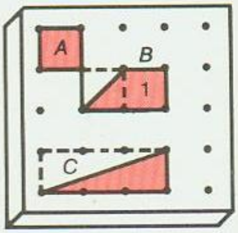
27. $ABCD$ is a trapezoid and E is the midpoint of $\overline{BC}$. Show that the area of $\triangle ADE$ is one half the area of $ABCD$.



28. Suppose $\triangle ABC$ is an equilateral triangle with altitude h. Let X be any point in the interior of the triangle, and let a, b, and c be the lengths of the perpendiculars to the sides of $\triangle ABC$. Prove that $a + b + c = h$. (*Hint:* Use area.)

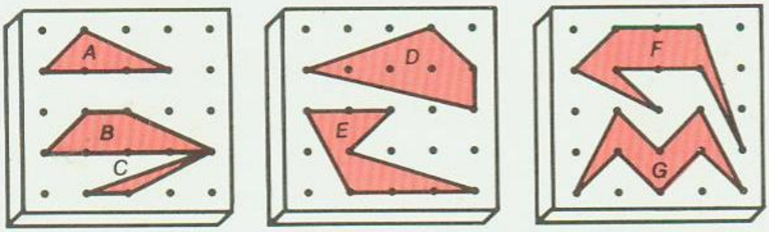


Activity

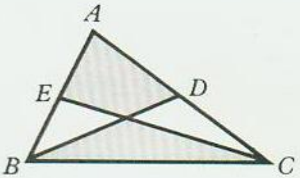


On this geoboard the area of square A is 1, and the area of each of figures B and C is $1\frac{1}{2}$.

1. Draw the figures below on dot paper or construct on a geoboard.
2. Find the area of each region below without using any area formulas.

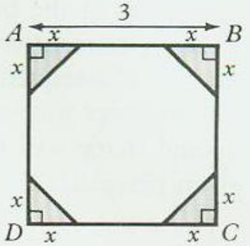


29. Given $\triangle ABC$ with medians $\overline{BD}$ and $\overline{CE}$. Show that the areas of the shaded regions are equal.



30. Prove that the area of an equilateral triangle whose sides have length s is $s^2\sqrt{3}/4$.

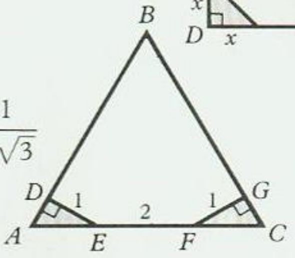
31. Consider the square $ABCD$. If triangles are cut off each corner, an octagon results. If $AB = 3$, show that $x = \dfrac{3}{2 + \sqrt{2}}$ if the resulting octagon is a regular octagon.



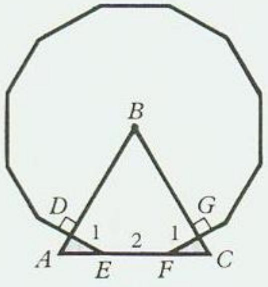
(Exs. 31, 32)

32. What is the area of a regular octagon whose side length is 2?

33. Consider the equilateral triangle $\triangle ABC$. Points D, E, F, and G are selected as shown so that $\overline{ED} \perp \overline{AB}$, $\overline{FG} \perp \overline{BC}$, $DE = FG = 1$, and $EF = 2$. Show that the area of $\triangle CFG$ is $\dfrac{1}{2\sqrt{3}}$ or $\dfrac{\sqrt{3}}{6}$.



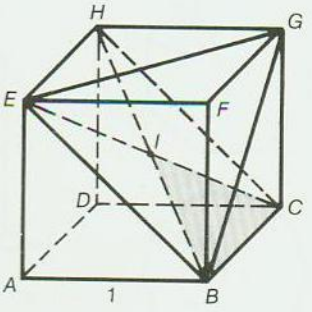
34. Show that the area of a regular 12-gon with side length 2 is $6(4 + 2\sqrt{3})$. (*Hint:* The figure on the right shows pentagon *BDEFG* is $\frac{1}{6}$ of a regular 12-gon, side length 2. Find the length *AC*, then use it to find the area of $\triangle ABC$. Then subtract the areas of triangles $\triangle ADE$ and $\triangle CFG$.)



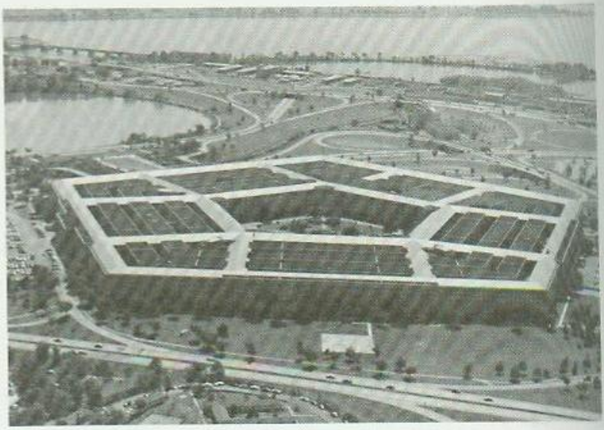
PROBLEM SOLVING

The length of the edges of the cube are all 1.

1. Find the length of $\overline{BE}$.
2. Find the length of $\overline{BH}$.
3. Find the area of $\triangle BEG$.
4. Find the area of rectangle *BCHE*.
5. Find the area of $\triangle BIC$.

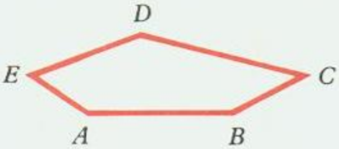


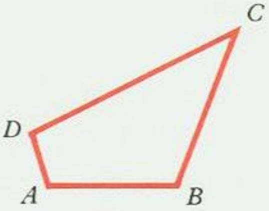
11–4 Area of Regular Polygons



The cost of construction for a building is influenced by the length of the outside walls—the perimeter of the building. A large perimeter requires more brick, siding, and window materials. Consequently, in designing a building, an architect might ask, "what regular polygonal shape will provide the most area for a given perimeter?"

There are two definitions needed.

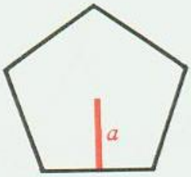


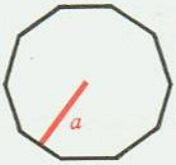


Definition 11-4

The **perimeter (p) of a polygon** is the sum of the lengths of the sides of the polygon.

$p = AB + BC + CD + DE + AE$ $\quad$ $p = AB + BC + CD + AD$

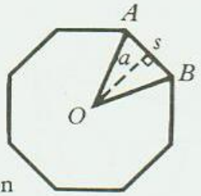




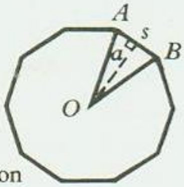
Definition 11-5

The **apothem (a) of a regular polygon** is the distance from its center to a side.

These two definitions are used to develop a formula for the area of a regular polygon with n sides. The table shown here helps analyze two examples.



regular 8-gon

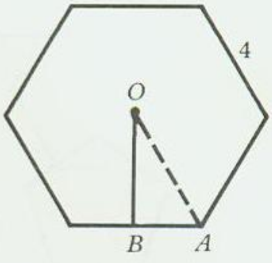


regular 10-gon

	Area of $\triangle ABO$	Perimeter (p)	Area of Polygon
8-gon (octagon)	$\frac{1}{2}as$	$p = 8s$	$8 \times \frac{1}{2}as = \frac{1}{2}a(8s)$ $= \frac{1}{2}ap$
10-gon (decagon)	$\frac{1}{2}as$	$p = 10s$	$10 \times \frac{1}{2}as = \frac{1}{2}a(10s)$ $= \frac{1}{2}ap$

Theorem 11–5 Given a regular n-gon with sides of length s and apothem a, the area A is given by the formula $A = \frac{1}{2}ans = \frac{1}{2}ap$ where perimeter $p = ns$.

Example



The side length of a regular hexagon is 4. Find the apothem and the area of the regular hexagon.

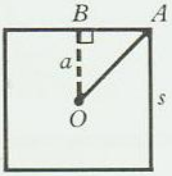
$\triangle OAB$ is a 30°-60°-90° triangle. Therefore,

$$AB = 2,\ OA = 4$$
$$a = OB = 2\sqrt{3}.$$

Applying Theorem 11–5, area $= \frac{1}{2}\underset{\underset{a}{\uparrow}}{(2\sqrt{3})} \cdot \underset{\underset{n}{\uparrow}}{6} \cdot \underset{\underset{s}{\uparrow}}{4} = 24\sqrt{3}.$

APPLICATION

If a square building and a regular hexagon building have the same perimeter (p), how do their areas compare?

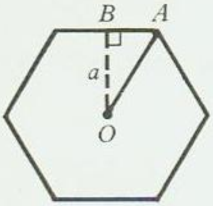


1. $\triangle OAB$ is a 45°-45°-90° triangle.

   Therefore the apothem $a = AB$

   $$= \frac{1}{2}s = \frac{1}{2}\left(\frac{p}{4}\right) = \frac{p}{8}.$$

   $$\text{Area of square} = \frac{1}{2} \cdot \frac{p}{8} \cdot p.$$



2. $\triangle OAB$ is a 30°-60°-90° triangle.

   The apothem $a = \sqrt{3}AB = \sqrt{3}(\frac{1}{2}s)$

   $$= \sqrt{3}\left(\frac{1}{2} \cdot \frac{p}{6}\right) = \frac{\sqrt{3}p}{12}.$$

   $$\text{Area of hexagon} = \frac{1}{2} \cdot \frac{\sqrt{3}}{12}p \cdot p.$$

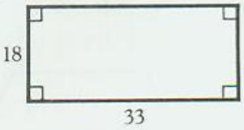
Since $\frac{\sqrt{3}}{2 \times 12} > \frac{1}{2 \times 8}$, the area of the hexagon is greater than the area of the square. Therefore a hexagonal building provides more area than a square building with the same perimeter.

EXERCISES

A.

For exercises 1–6, find the perimeter of the given figure.

1.



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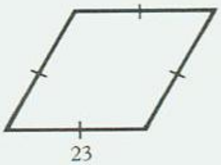
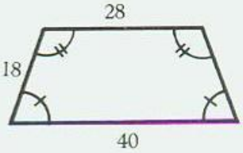


3.



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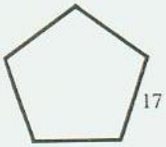

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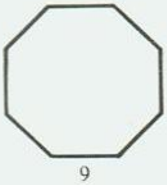

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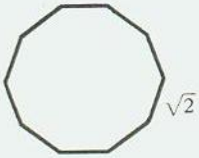
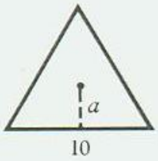


For exercises 7–9, find the apothem and the area of the given regular polygon.

7.



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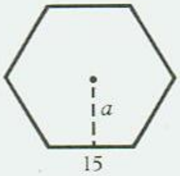

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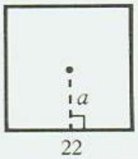
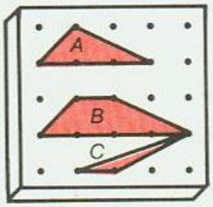


Activity

In the activity on page 406 you found the area of some regions on a geoboard without using any area formulas. Complete the table below. Look for a formula for the area of these polygons in terms of the number of nails on the boundary of the figure (*d*) and the number of interior points (*i*).



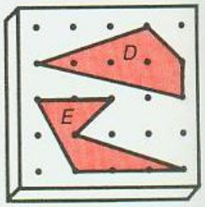
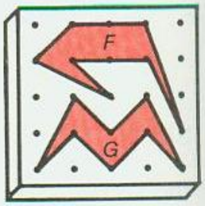


Figure	Number of nails on boundary (*d*)	Number of interior points (*i*)	$\frac{d}{2} + i$	Area of polygon
A	?	?	?	?
B	7	0	$\frac{7}{2} + 0$	$\frac{5}{2}$
C	?	?	?	?
⋮				



B.

10. Find the area of a regular hexagon with apothem $3\sqrt{3}$.

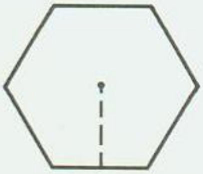
11. Find the area of a regular octagon with side length 5 and apothem k.

12. If the area of a regular hexagon is $36\sqrt{3}$ cm^2, what is its apothem and the length of its side?

13. If the apothem of a regular hexagon is 5 m, what is its perimeter? Its area?

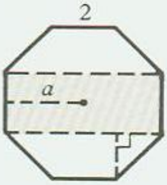
C.

14. If an equilateral triangle and a regular hexagon have equal perimeters, prove that the ratio of their areas is 2 to 3.



15. The area of a regular hexagon is $50\sqrt{3}$ sq ft. What is its perimeter? Its apothem?

16. The side length of a regular octagon is 2. What is its apothem?

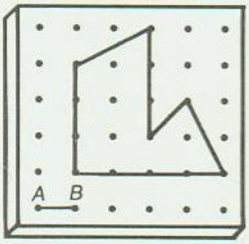


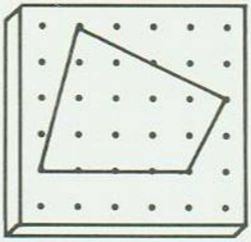
17. A farmer is to enclose a rectangular pen with 100 meters of fence. He is trying to decide what shape the pen should have. Fill out this table and see if you have any recommendation for the farmer.

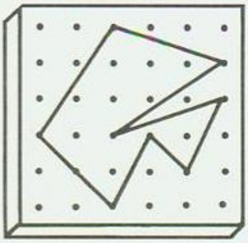
length	width	perimeter	area
48 m	?	100 m	?
45 m	?	100 m	?
40 m	?	100 m	?
35 m	?	100 m	?
30 m	?	100 m	?
25 m	?	100 m	?

PROBLEM SOLVING

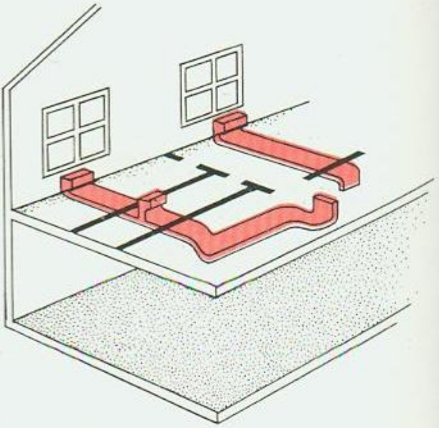
Find the perimeter of each polygon on these geoboards. Note that $AB = 1$.







11–5 Comparing Perimeters and Areas of Similar Polygons



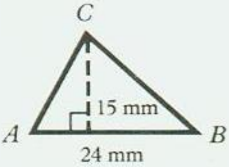
An engineer designing the heating-cooling system for a building must be able to answer the following types of questions.

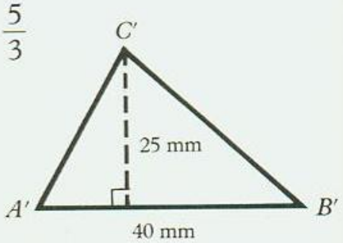
1. How much more air can be moved by a 7-inch × 14-inch duct than by a 5-inch × 10-inch duct?
2. Which is more economical, running two 5-inch × 10-inch ducts or one 7-inch × 14-inch duct?

These questions require a comparison between the cross-sectional areas of a 5×10 rectangle and a 7×14 rectangle—a pair of similar polygons.

Study these two examples. We are comparing the areas of a pair of similar triangles and a pair of similar hexagons.

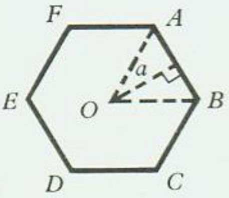
Example 1 $\triangle A'B'C' \sim \triangle ABC$ with $\dfrac{A'B'}{AB} = \dfrac{A'C'}{AC} = \dfrac{B'C'}{BC} = \dfrac{5}{3}$

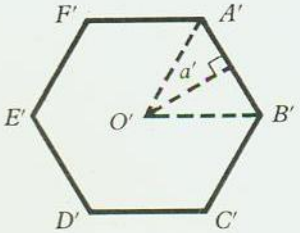




$$\frac{\text{perimeter } \triangle A'B'C'}{\text{perimeter } \triangle ABC} = \frac{5}{3} \qquad \frac{\text{area } \triangle A'B'C'}{\text{area } \triangle ABC} = \frac{\frac{1}{2}(40)(25)}{\frac{1}{2}(24)(15)} = \frac{500}{180} = \frac{25}{9} = \left(\frac{5}{3}\right)^2$$

Example 2 $A'B'C'D'E'F' \sim ABCDEF$ with $\dfrac{A'B'}{AB} = \dfrac{2}{1}$





$$\frac{\text{perimeter } (A'B'C'D'E'F')}{\text{perimeter } (ABCDEF)} = \frac{2}{1} \qquad \frac{\text{area } (A'B'C'D'E'F')}{\text{area } (ABCDEF)} = \frac{6 \cdot A(\triangle A'O'B')}{6 \cdot A(\triangle AOB)} = \frac{\frac{1}{2}a'A'B'}{\frac{1}{2}aAB} = \left(\frac{2}{1}\right)^2$$

We summarize these examples in this table.

	Ratio of Corresponding Sides	Ratio of Perimeters	Ratio of Areas
Example 1	$\frac{A'B'}{AB} = \frac{40}{24} = \frac{5}{3}$	$\frac{5}{3}$	$\left(\frac{5}{3}\right)^2$
Example 2	$\frac{A'B'}{AB} = \frac{2}{1}$	$\frac{2}{1}$	$\left(\frac{2}{1}\right)^2$

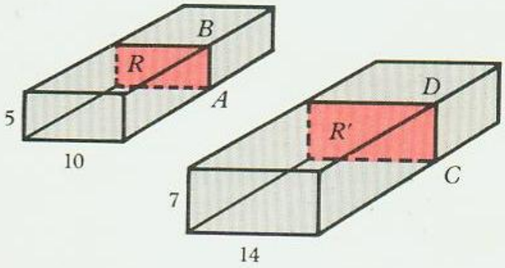
These examples suggest the following theorems.

Theorem 11–6 The ratio of the perimeters of two similar polygons is equal to the ratio of the lengths of any pair of corresponding sides.

Theorem 11–7 The ratio of the areas of two similar polygons is equal to the square of the ratio of the lengths of any pair of corresponding sides.

APPLICATION

We shall answer the two questions asked at the beginning of the lesson by comparing the area and perimeter of the rectangular cross sections of the two ducts.



$$\frac{CD}{AB} = \frac{7}{5} \quad \frac{\text{area}(R')}{\text{area}(R)} = \left(\frac{7}{5}\right)^2 = \frac{49}{25} \doteq \frac{2}{1} \quad \frac{\text{perimeter}(R')}{\text{perimeter}(R)} = \frac{7}{5}$$

Conclusion: The cross-section area (and hence the amount of air moved) is nearly twice as great for the larger duct. Yet its cross-section perimeter is only $\frac{7}{5}$ as great as the smaller duct. Consequently less material would be required to build it than two of the smaller ducts.

EXERCISES

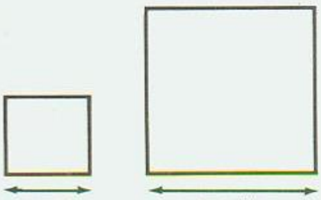
A.

Suppose the lengths of the sides of two squares are 4 and 8 respectively.

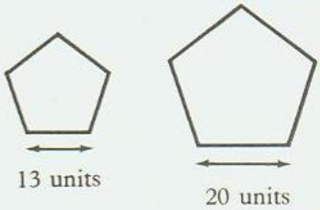
1. What is the ratio of their perimeters?
2. What is the ratio of their areas?

Two regular pentagons have side lengths in a ratio of 13:20.

3. What is the ratio of the perimeters of these pentagons?
4. What is the ratio of the areas of these two pentagons?



(Exs. 1,2)



(Exs. 3,4)

B.

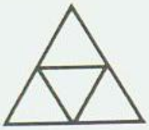
5. The ratio of the perimeters of two similar polygons is $\frac{\sqrt{3}}{2}$. What is the ratio of their areas?
6. If the ratio of the areas of two similar n-gons is $\frac{36}{25}$, what is the ratio of their perimeters?
7. The ratio of the areas of two similar polygons is $\frac{25}{9}$ and the sum of the two areas is 272. Find the areas of the two polygons.

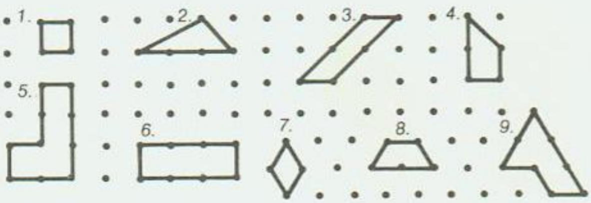
Activity

An equilateral triangle is called a "reptile" (an abbreviation for "repeating tile") because four equilateral triangles can be arranged to form a larger equilateral triangle. Which of these figures are "reptiles?"

Answer this question by either drawing on dot paper or cutting out four copies of each figure and trying to arrange them.

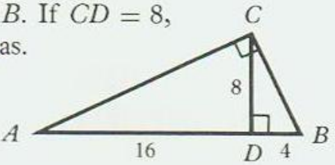
A "reptile"



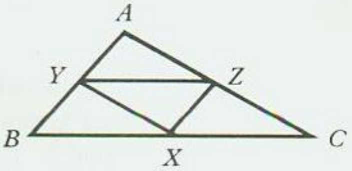


8. Suppose $\triangle ABC$ is a right triangle and $\overline{CD} \perp \overline{AB}$. If $CD = 8$, $AD = 16$, and $BD = 4$, find these ratios of areas.

a. $\dfrac{A(\triangle ACD)}{A(\triangle CBD)}$ **b.** $\dfrac{A(\triangle ACD)}{A(\triangle ABC)}$

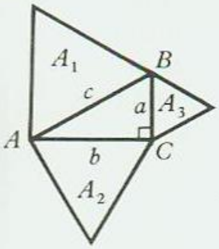


C.

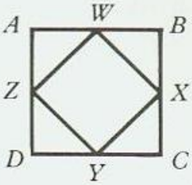


9. Suppose points X, Y, and Z are the midpoints of the sides of $\triangle ABC$. Find the ratio $A(\triangle XYZ):A(\triangle ABC)$.

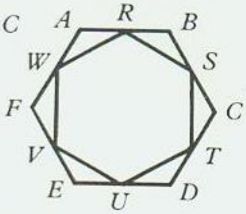
10. Suppose $\triangle ABC$ is a right triangle with hypotenuse c and legs a and b. Construct equilateral triangles on the sides of $\triangle ABC$ as shown. If the areas of these triangles are A_1, A_2, and A_3 as shown, show that $\dfrac{A_2}{A_1} + \dfrac{A_3}{A_1} = 1$.



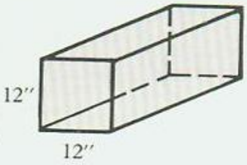
11. Points W, X, Y, and Z are midpoints of the sides of square $ABCD$. Find $\dfrac{A(ABCD)}{A(WXYZ)}$.



12. Points R, S, T, U, V, and W are midpoints of the sides of regular hexagon $ABCDEF$. Find $\dfrac{A(ABCDEF)}{A(RSTUVW)}$.



13. A heating contractor made a duct with a square cross section 12 inches on a side. How long should one side of a duct be that has a cross section that carries twice as much air?

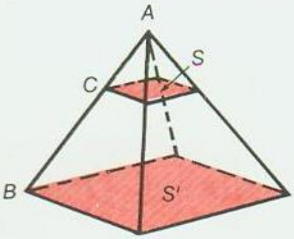


14. If the amount of air carried by a 10-inch square duct is to be increased by 30%, what size square duct should be used? (Round off to the nearest $\frac{1}{2}$ inch.)

PROBLEM SOLVING

A square-based pyramid is sliced by a plane passing through edge $\overline{AB}$ and point C. If the plane is parallel to the square base, a square cross section results.

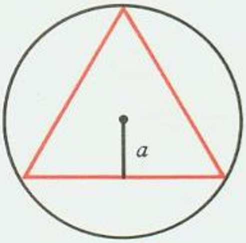
If $\dfrac{AC}{CB} = \dfrac{1}{3}$, find $\dfrac{A(S)}{A(S')}$.

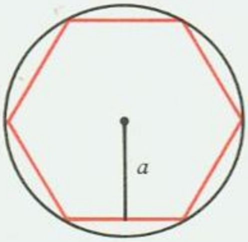


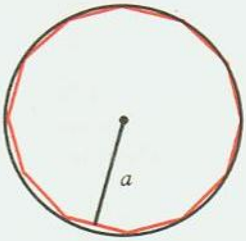
11–6 Ratio of Circumference and Diameter of a Circle

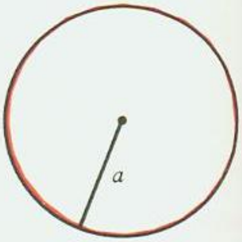
In computer graphics the plotter of the computer draws what appears to be curved lines. Actually the pen of the plotter draws successive straight line segments that are so short that the overall effect is a curved line. This same concept is used in finding the circumference of a circle.

These figures show a sequence of regular polygons that get closer and closer to a circle.









As the number of sides of the regular polygon increases, the polygon is nearly identical to its circumscribed circle. Also, the perimeter becomes close to a fixed number called the *circumference* of the circle, and the apothem becomes close to the radius of the circumscribed circle.

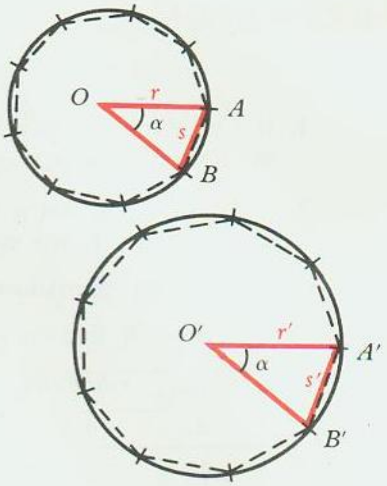
The following theorem is basic to the theory of circles.

Definition 11-6

The **circumference of a circle** is the number approached by the perimeters of the inscribed regular polygons as the number of sides of the regular polygons increases.

Theorem 11–8 The ratio of the circumference to the diameter is the same for all circles.

OUTLINE OF PROOF



1. Select any two circles and inscribe in each a regular n-gon.
2. A pair of similar isosceles triangles $\triangle AOB$ and $\triangle A'O'B'$ are determined by one side of each n-gon.
3. Therefore, the ratios $\frac{s}{r}$ and $\frac{s'}{r'}$ are equal and $\frac{ns}{r} = \frac{n's'}{r'}$.
4. The numbers ns and $n's'$ equal the perimeters p and p' of the two regular n-gons. Therefore,
$$\frac{p}{r} = \frac{p'}{r'}.$$
5. As the number of sides n increases, the perimeters p and p' approach the circumferences C and C'. Therefore
$$\frac{C}{r} = \frac{C'}{r'} \text{ and } \frac{C}{2r} = \frac{C'}{2r'}.$$

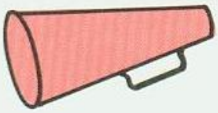
The ratio $\frac{C}{d}$ is an irrational number, which means it cannot be written exactly as a decimal. Some approximations of this number are 3.14, $3\frac{1}{7}$, 3.14159.

Definition 11-7

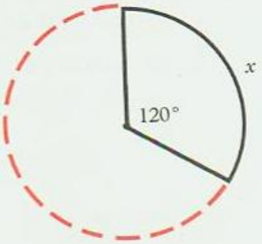
The **ratio** $\frac{C}{d}$, which is the same real number for any circle, is denoted by π (the Greek letter pi).

Theorem 11-9 Given a circle with radius r and diameter d, the circumference C is given by the formula $C = \pi d = 2\pi r$.

APPLICATION



A cheerleader wants to make a pattern for a megaphone. This pattern is a portion of a circle bounded by a central angle and its intercepted arc. If the sides of the pattern are 15 inches and the central angle has a measure of 120°, find the length of the intercepted arc.



Solution. We can set up a proportion between the arc length, the circumference of the circle, the measure of the central angle, and the degree measure of the circle.

$$\frac{120}{360} = \frac{x}{\text{circumference}} = \frac{x}{2\pi 15}$$

$$\frac{1}{3} = \frac{x}{30\pi \text{ in.}} \quad \text{or} \quad x = 10\pi \doteq 31.4 \text{ in.}$$

EXERCISES

A.

1. If a regular 100-gon is inscribed in a circle, the perimeter of the 100-gon is nearly equal to the _?_ of the circle.

2. If a regular 100-gon is inscribed in a circle, the apothem of the 100-gon is nearly equal to the _?_ of the circle.

3. Give several approximations for the value π.

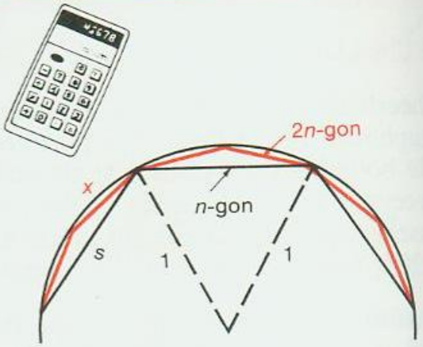
For exercises 4–8, find the missing numbers.

	radius	*diameter*	*circumference*
4.	2	?	4π
5.	?	6	?
6.	$7/\pi$	?	?
7.	?	?	8π
8.	?	?	16

B.

9. In a circle what does the ratio $\frac{C}{\pi}$ equal?

Activity



Finding an approximate value of π. (Use a calculator.) The following formula can be proved.

$x = \sqrt{2 - \sqrt{4 - s^2}}$

s = length of side of the regular n-gon inscribed in a circle of radius 1.

x = length of a side of the corresponding $2n$-gon.

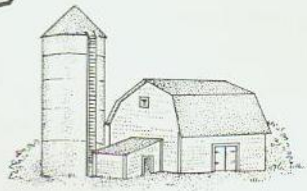
Complete this table and approximate the value of π.

Continue the table for $n = 48, 96, 192, 384$, and 768.

Number of sides (n)	Length of side (s)	Perimeter ($n \cdot s$)	Perimeter ÷ diameter	Side length of $2n$ gon (x)
6	1	6	3	$\sqrt{2 - \sqrt{4 - (1)^2}} = 0.517638$
12	0.517638	6.211656	3.105828	$\sqrt{2 - \sqrt{4 - (.517638)^2}} = 0.261052$
24	0.261052	?	?	

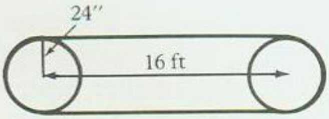
10. Find the length of an arc intercepted by a 60° central angle in a circle with radius 10. (Leave your answer in terms of π.)

11. A rectangular piece of cardboard is bent to form a tube 12 inches long and 3 inches in diameter. What is the area of this piece of cardboard?

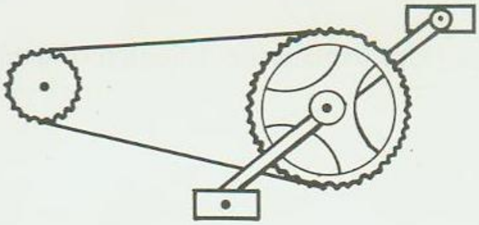


12. If a gallon of paint covers 400 square feet, how many gallons are required to paint a silo (excluding the roof) that is 10 feet in diameter and 50 feet high?

13. On a large machine the centers of two pulleys are 16 feet apart and the radius of each pulley is 24 inches. How long a belt is needed to wrap around both pulleys?



14. The large sprocket on the pedals of a bicycle has 50 teeth and the small sprocket on the rear wheel has 20 teeth. When the pedals make two complete revolutions, how many revolutions does the wheel make?

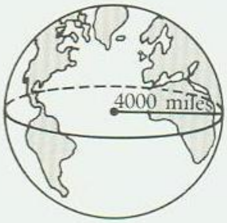


C.

15. A round tower with a 10-meter circumference is surrounded by a fence that is 2 meters from the tower. How long is the fence?

16. How far does a bicycle travel with each 25 revolutions of a wheel if the outside diameter of a tire is 29 inches?

17. Assume the earth's equator is a perfect circle with radius 4000 miles, and that a rope is tied tightly around the equator. Suppose a 40-foot piece of rope has been spliced into the rope and the rope has been propped up to form a fence. What would be the distance between the fence and the earth?

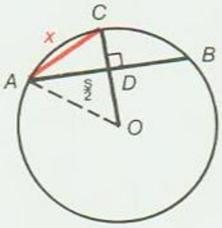


PROBLEM SOLVING

Use the Pythagorean Theorem twice to prove the formula that you used in the Activity on the opposite page.

Given: $OC = 1$
$AB = s$, $AC = x$
$AD = \frac{s}{2}$

Find: OD, CD, and x.

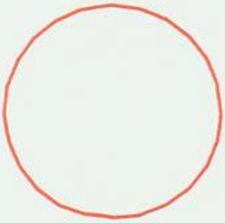


11–7 Area of Circles

A city building inspector must be sure that the water mains are large enough to meet the water demand in each part of the building. How much more water does a 6-inch water main carry than a 4-inch water main?

To show that the greater volume is $2\frac{1}{4}$ times the lesser volume, you must compare the area of the circular cross section of the larger pipe to the area of the circular cross section of the smaller pipe.

This figure helps explain the definition of area of a circle.

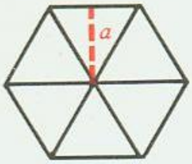


The area of an inscribed regular n-gon is a good approximation of the area of the circumscribed circle when n has a large value.

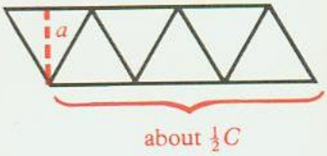
Definition 11-8

The **area of a circle** is the number approached by the areas of the inscribed regular n-gons as n gets larger and larger.

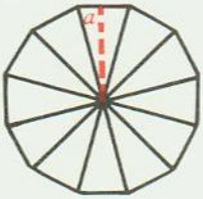
An inscribed regular polygon can be cut into triangles, which can be rearranged to form a parallelogram.

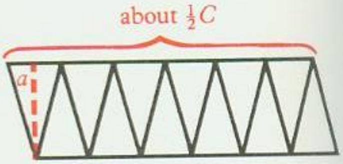


The area of the parallelogram is closely approximated by $\frac{1}{2}Ca$, $\frac{1}{2}(2\pi r)a$, or πra.

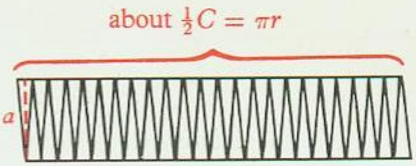


This approximation improves when the number of sides of the regular polygon is increased.





As the number of sides increases, the number of triangles making up the parallelogram increases.

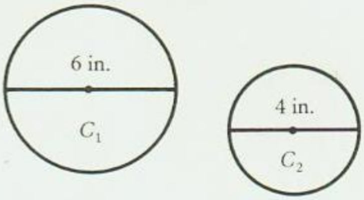


The apothem a approaches the radius r and the area πra approaches πr^2.

Theorem 11-10 Given a circle with radius r, the area A is given by the formula $A = \pi r^2$.

APPLICATION

The amount of water that a 6-inch pipe can carry can be compared to the water carried by a 4-inch pipe. We can do this by forming a ratio of the areas.

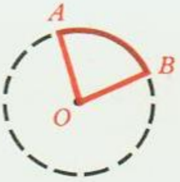


$A(C_1) = \pi(3)^2 = 9\pi$

$A(C_2) = \pi(2)^2 = 4\pi$

$\frac{A(C_1)}{A(C_2)} = \frac{9\pi}{4\pi} = 2.25$ or $2\frac{1}{4}$ times as much.

The formula for the area of a circle can be used to find the area of a region called a sector.



AOB is a sector of $\odot O$.

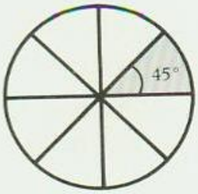
Definition 11-9

A **sector** is a region bounded by a central angle and its intercepted arc.

The ratio $\frac{\text{area of a sector}}{\text{area of the circle}}$ is equal to the ratio $\frac{\text{degree measure of the central angle}}{360^\circ}$.

APPLICATION

If a 16-inch pizza is cut into eight congruent pieces, what is the area of one piece?



$\frac{\text{area of one piece}}{64\pi \text{ sq in.}} = \frac{45^\circ}{360^\circ}$

Area(one piece) $= \frac{1}{8} \times 64\pi$ sq in. $= 8\pi$ sq in.

EXERCISES

A.

For exercises 1–4, express the answers as an exact number (use the number π).

1. Find the area of a circle with radius

a. 2 **b.** $5\frac{1}{2}$ **c.** π **d.** $\sqrt{3}$

2. Find the area of a circle with diameter

a. 6 **b.** $7\frac{1}{2}$ **c.** 3π **d.** $4\sqrt{2}$

3. Find the area of a circle whose circumference is

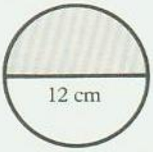
a. 2π **b.** 6π **c.** $\sqrt{6}\pi$ **d.** 10

4. Find the radius of a circle whose area is

a. 144π **b.** 225π **c.** 12π **d.** 100

For exercises 5–8, find the area of the shaded sectors. Give answers in terms of π.

5.



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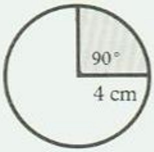

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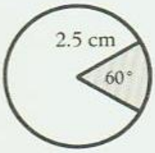
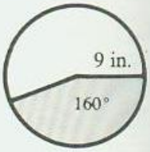


8.



9. Find the approximate area of the following sectors. Use 3.14 for π.

a. Central angle, 50°
Radius, 3 cm

b. Central angle, 75°
Radius, 3 m

c. Central angle, 15°
Radius, 10 in.

10. If the area of a sector is one tenth the area of the circle, what is the central angle of the sector?

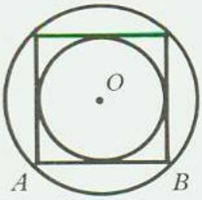
B.

11. Two circles have radii of 4 cm and 5 cm, respectively. What is the ratio of their areas?

12. The areas of two circles have a ratio of 9 to 4. What is the ratio of their radii?

13. The areas of two circles have a ratio of 8 to 5. What is the ratio of their radii?

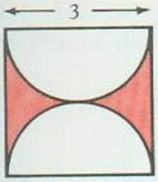
14. Find the area of the inscribed and circumscribed circles of a square whose side is 4 cm.



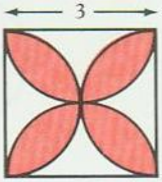
$AB = 4$ cm

For exercises 15–17, find the area of the shaded region. The length of one side of the square is 3.

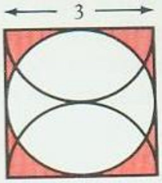
15.



16.

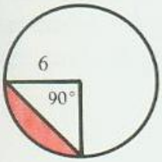


17.

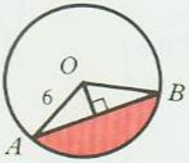


The shaded areas in the figures for exercises 18–20 are called **segments of the circle.** Find the area of each segment of the circle.

18.

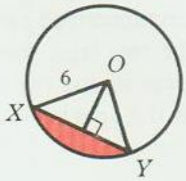


19.



$m\angle AOB = 120$

20.

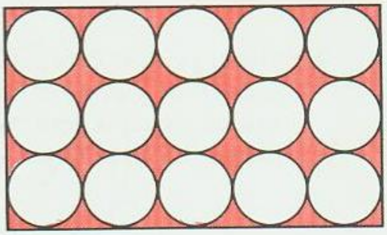


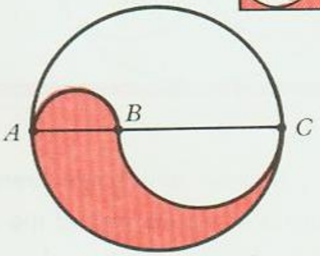
$m\angle XOY = 60$

C.

21. Circles of equal radii are packed in a rectangle as shown. What fraction of the rectangular region is shaded?

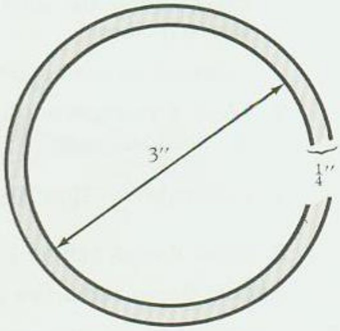
22. A pipe 12 inches in diameter carries how many times as much water as a pipe 10 inches in diameter?



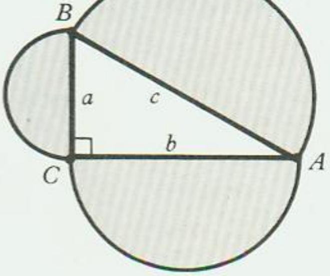


23. If $BC = 2AB$, what fraction of the circle is shaded?

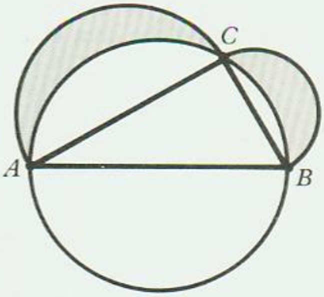
24. The figure at the right represents the cross section of a pipe $\frac{1}{4}$ inch thick that has an inside diameter of 3 inches. Find the area of the shaded region.



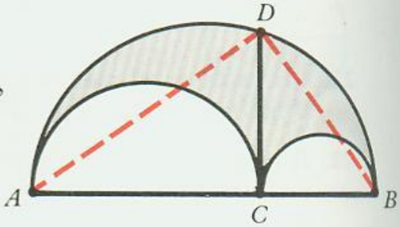
25. Given a right triangle $\triangle ABC$, show that the area of a semicircle on the hypotenuse is equal to the sum of the areas of the semicircles on the two legs of the triangle.



26. Given a right triangle $\triangle ABC$, its circumscribed circle, and semicircles on the two legs, show that the sum of the areas of the two shaded regions is equal to the area of $\triangle ABC$.



27. Given point C between A and B, and semicircles on $\overline{AC}$, $\overline{CB}$, and $\overline{AB}$ as shown. If $\overline{CD} \perp \overline{AB}$, show that the area of the shaded region is equal to the area of the circle with $\overline{CD}$ as diameter. (*Hint:* Consider the right triangle $\triangle ADB$.)



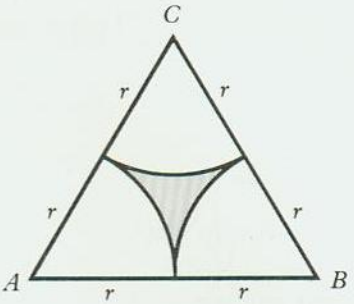
Activity

Estimate the value of π through direct measurement.

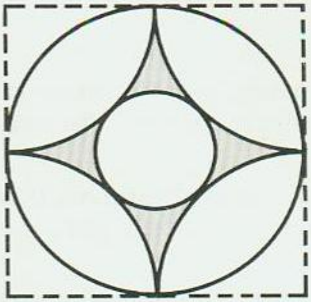
1. Select a circular object about the size of the top of a round waste basket.
2. Measure its diameter (d) to the nearest millimeter.
3. Wrap a string around the object and measure its length to find its circumference (C) to the nearest millimeter.
4. Calculate $\frac{C}{d}$. How accurate is your estimate of π?
5. What things can you do to improve your estimate?

Repeat these same five steps using a tin can and a bicycle wheel.

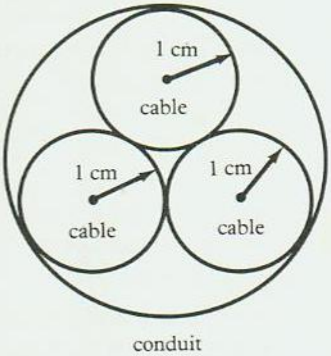
28. If $\triangle ABC$ is an equilateral triangle, what fraction of the triangle is shaded?



29. In the figure the small circle is tangent to four circular arcs. What fraction of the large circle is shaded?



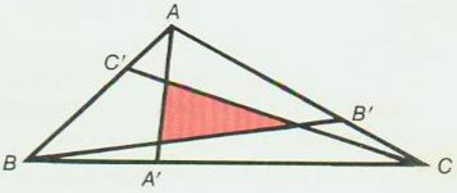
30. Conduit for telephone cable is sized so that it carries three cables (each circular and tangent to the conduit and to each other) that are each 1 cm in radius. What fraction of the conduit is filled?



PROBLEM SOLVING

Points A', B', and C' are points of trisection of the sides of $\triangle ABC$. The area of the shaded triangular region is $\underline{?}$ the area of $\triangle ABC$.

Use a grid of small squares or a measurement technique to decide whether the blank should be filled in with the fraction $\frac{1}{5}$. $\frac{1}{6}$, $\frac{1}{7}$, or $\frac{1}{8}$.



Important Ideas—Chapter 11

Terms

Polygonal region (p. 394)
Square unit (p. 398)
Altitude of a parallelogram (p. 399)
Perimeter of a polygon (p. 408)
Apothem of a regular polygon (p. 408)
Circumference of a circle (p. 416)
Pi(π) (p. 417)
Area of a circle (p. 420)
Sector (p. 421)
Segment of a circle (p. 423)

Postulates

Area Postulate. A unique positive number called the area can be assigned to each polygonal region. The area of region R is denoted by $A(R)$.

Area of Congruent Regions Postulate. If two rectangles or two triangles are congruent, then the regions they bound have the same area.

Area Addition Postulate. If a polygonal region is the union of n non-overlapping polygonal regions, then its area is the sum of the areas of these n-regions.

Rectangle Area Postulate. The area of a rectangle with length ℓ and width w is given by the formula ℓw.

Theorems

11-1 Given a parallelogram with base b and corresponding height h, the area A is given by the formula $A = bh$.

11-2 Given a triangle with base b and corresponding altitude h, the area A is given by the formula $A = \frac{1}{2}bh$.

11-3 If $\triangle ABC$ has sides of length a, b, and c, then $A(\triangle ABC) = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$.

11-4 Given a trapezoid with bases b_1 and b_2 and altitude h, the area A is given by the formula $A = \frac{1}{2}h(b_1 + b_2)$.

11-5 Given a regular n-gon with sides of length s and apothem a, the area A is given by the formula $A = \frac{1}{2}ans = \frac{1}{2}ap$ where perimeter $p = ns$.

11-6 The ratio of the perimeters of two similar polygons is equal to the ratio of the lengths of any pair of corresponding sides.

11-7 The ratio of the areas of two similar polygons is equal to the square of the ratio of the lengths of any pair of corresponding sides.

11-8 The ratio of the circumference to the diameter is the same for all circles.

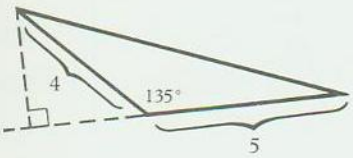
11-9 Given a circle with radius r and diameter d, the circumference C is given by the formula $C = \pi d = 2\pi r$.

11-10 Given a circle with radius r, the area A is given by the formula $A = \pi r^2$.

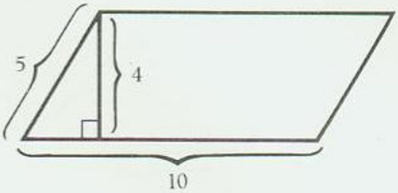
Chapter 11—Review

1. Find the area of the figures below. Assume that segments are parallel or congruent if they appear so.

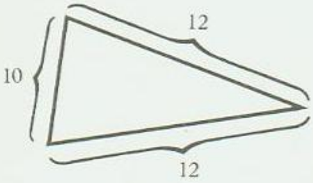
a.



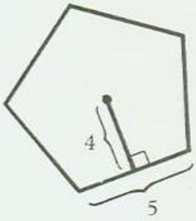
b.



c.



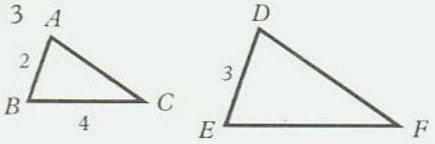
d.



2. **Given:** $\triangle ABC \sim \triangle DEF$, area$(\triangle ABC) = 3$

   a. Find the length of the altitude to $\overline{AB}$.

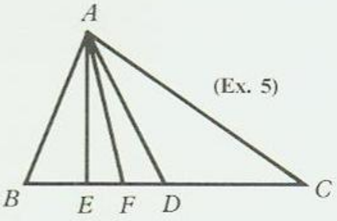
   b. Find the area of $\triangle DEF$.



3. The circumference of circle O' is twice that of circle O. What is the ratio of the lengths of their diameters?

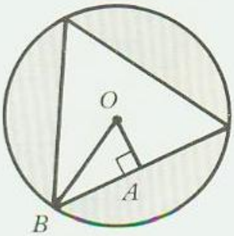
4. Find the area of a regular hexagon inscribed in a circle with a diameter of 6 cm.

5. In $\triangle ABC$, $\overline{AE}$ is an altitude, $\overline{AF}$ is an angle bisector, and $\overline{AD}$ is a median. Which segment divides $\triangle ABC$ into two triangles of equal area?



(Ex. 5)

6. Find the area of the shaded portions. Assume segments are congruent if they appear so.

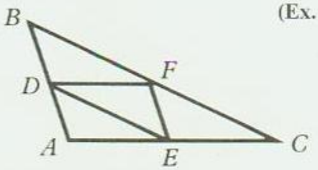


(Ex. 6)

$OA = 1$
$OB = 2$

7. In $\triangle ABC$, D, E, and F are midpoints.

   Find the ratio $\dfrac{\text{area}(\triangle DEF)}{\text{area}(\triangle ABC)}$.

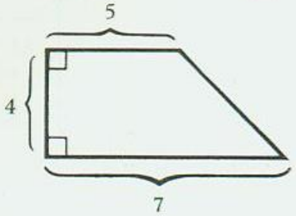


8. Find the area of a circle that is inscribed in a square whose area is 16 square units.

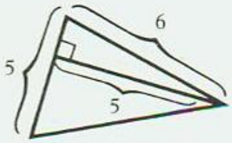
Chapter 11—Test

1. Find the area of the figures below. Assume that segments are parallel or congruent if they appear so.

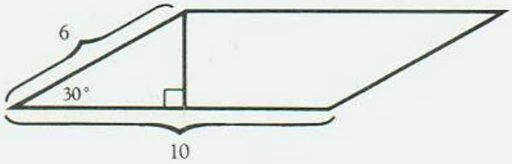
a.



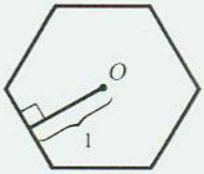
b.



c.



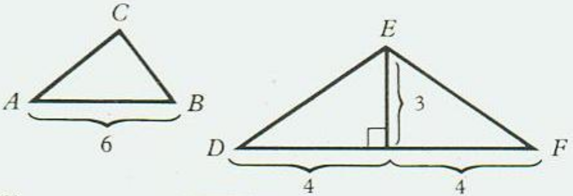
d.



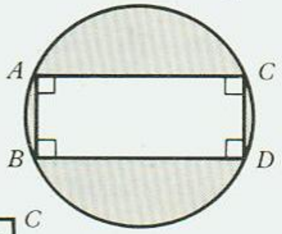
2. **Given:** $\triangle ABC \sim \triangle DFE$

   a. Find the perimeter of $\triangle ABC$.

   b. Find the area of $\triangle ABC$.



3. Find the area of the shaded portions, where $AB = 10$ and $BC = 26$.

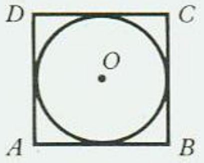


4. If each side of a regular polygon is doubled, how is the perimeter affected? How is the area affected?

5. Find the following two ratios.

   a. $\dfrac{\text{perimeter of square } ABCD}{\text{circumference of circle } O}$

   b. $\dfrac{\text{area}(ABCD)}{\text{area}(\odot O)}$

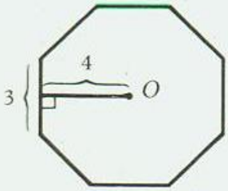


   c. Do the answers to **a** and **b** depend on the size of the figure?

6. If two triangles are similar and their perimeters are in a ratio of 2:1, what is the ratio of their areas?

7. Find the circumference and area of a circle with diameter of 4 cm. How are they different?

8. Find the area of the regular polygon.



(Ex. 8)

Algebra Review

Evaluate each formula for the letter indicated.

1. $A = bh$; for A if $b = 6$ cm, $h = 4$ cm
2. $A = \frac{1}{2}h\,(b_1 + b_2)$; for A if $h = 7$ cm, $b_1 = 9$ cm, $b_2 = 15$ cm
3. $C = 2\pi r$; for C if $r = 14$ cm (use $\pi \doteq 3.14$).
4. $A = 4\pi r^2$; for A if $r = 14$ cm
5. $V = \frac{4}{3}\pi r^3$; for V if $r = 14$ cm
6. $V = \frac{1}{3}Bh$; for h if $V = 100\text{ cm}^3$, $B = 30\text{ cm}^2$
7. $A = \frac{1}{2}h\,(b_1 + b_2)$; for h if $A = 36\text{ cm}^2$, $b_1 = 12$ cm, $b_2 = 8$ cm
8. $A = 4\pi r^2$; for r if $A = 100\pi\text{ m}^2$
9. $A = \pi r\ell + \pi r^2$; for A if $r = 4\frac{1}{2}$ in., $\ell = 8\frac{1}{2}$ in.
10. $A = \dfrac{s^2\sqrt{3}}{4}$; for s if $A = 6\sqrt{3}$ sq yd

Solve for x.

11. $(x + 2)(x - 3) = 0$
12. $x^2 - 9 = 0$
13. $x^2 + 4x + 4 = 0$
14. $x^2 - 6x + 9 = 0$
15. $x^2 + 5x = -6$
16. $x^2 + x - 2 = 0$
17. $x(x - 1) = 90$
18. $6x^2 - 7x = 5$
19. $1 - 8x + 15x^2 = 0$
20. $x^2 - 11x = 180$
21. $x(x - 5) + 6 = 0$
22. $0 = 5x - 3x^2 + 2$

Solve.

23. Find the area of a rectangular field whose length is 100 yards and width is 79 yards.
24. Find the area of a square whose perimeter is 20 cm.
25. The perimeter of a rectangle is 40 cm and its length is 5 cm more than its width. Find its area.
26. A circular pond has a diameter of 48 meters. What is its area?
27. The base of a triangle is 3 times as long as its height. If together they measure 72 mm, what is the area of the triangle?
28. The length and width of a rectangle together measure 100 yards. Their difference is 7 yards. What is the area of the rectangle?
29. The dimensions of a rectangular flower garden are 40 meters by 24 meters. There is a walk around the garden. The area of the garden and the walk is 1232 square meters. Find the width of the walk.
30. The width of a rectangle is 16 cm. The diagonal is 4 cm more than the length. Find the length of the rectangle.

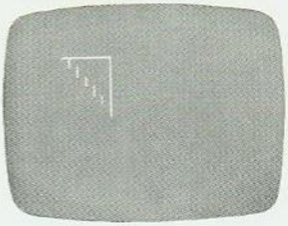
Geometry In Our World

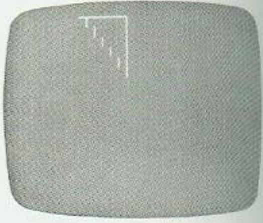
Computer Graphics: Transformations

An important use of computer graphics is the analysis of shapes. First we will discuss the idea of moving a shape in a plane. The computer can be programmed to move a figure to different positions on the screen. These movements are called transformations.

One of the transformations that may be used is the translation. In a program the following command will cause the point (x, y) to be moved 50 units to the right and 70 units up. It is one line in a program that causes a figure to be translated.

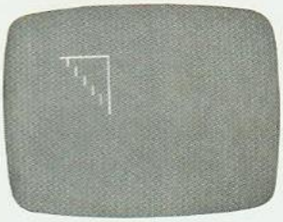
```
HPLOT X,Y TO X+50, Y+70
```

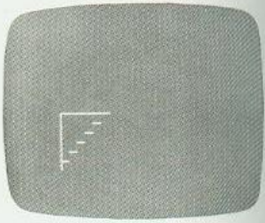

Before translation


After translation

In a program the following command will cause a point (x, y) to be rotated 90° counterclockwise about the origin. It could be one line in a program that causes a figure to be rotated.

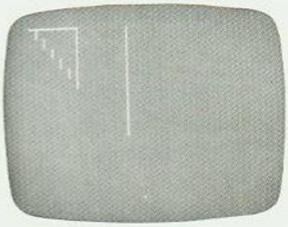
```
HPLOT X, Y TO -Y,X
```

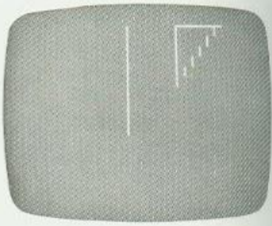

Before rotation


After rotation

In a program the following command will cause a point (x, y) to be reflected in the y-axis. It could be one line in a program that causes a figure to be reflected.

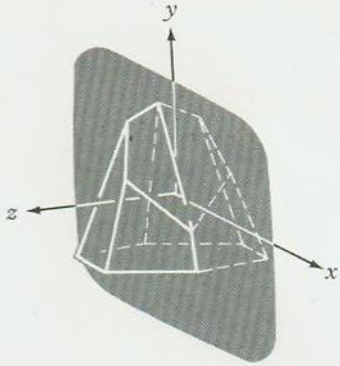
```
HPLOT X,Y TO -X,Y
```


Before reflection

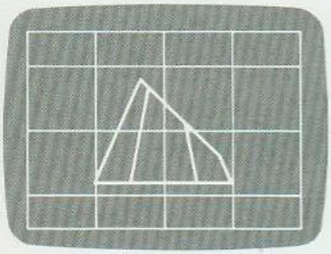

After reflection

The benefit for using the computer to show different views of an object is particularly great for 3-dimensional solids. Here we see an example of such a solid.

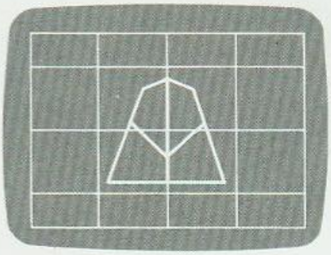
Below we see how a computer can help show different views of this solid.



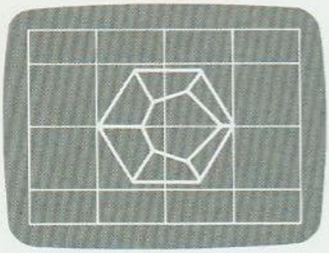
In each of these figures the computer screen shows the solid as you look at the *xy*-plane.



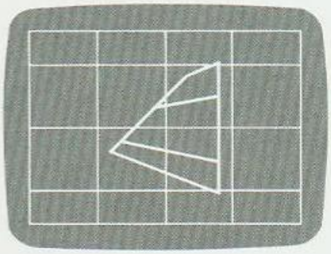
A view onto the *xy*-plane



A view after a 90° rotation about the *y*-axis



A view after a 90° rotation about the *x*-axis



A view after a 90° rotation about the *z*-axis

1. Sketch the same four views as shown above for the solid shown here.
2. Draw a 3-dimensional sketch of your own. Draw the same four views as shown above for your solid.